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Volume expansion rate of the Lorentzian manifold based on integral Ricci curvature over a timelike geodesic

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Abstract

Recent astronomical observations show that the universe is not only expanding but also undergoing accelerated expansion [A.G. Riess, et al., The farthest known supernova, Astrophys. J. 560 (2001) 49–71; P.K. Townsend, M.N.R. Wohlfarth, Accelerating cosmologies from compactification, Phys. Rev. Lett. 91 (2003) 061302]. Then the timelike convergence condition does not hold every time, i.e. the Ricci curvature $\mathrm{Ric}(v, v)$ cannot be nonnegative for every timelike vector v. We obtain the volume expansion rate of the universe based on the integral norm of negative part of the Ricci curvature along a timelike geodesic.

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1. Introduction

Let M be an n-dimensional Lorentzian manifold. In classical general relativity, it is assumed that the Ricci curvature satisfies that

$$\mathrm{Ric}(v, v) \geq 0$$

for any timelike unit vector v, which is the timelike convergence condition, which says that *gravity attracts on average* [4]. Then the timelike convergence condition implies that there can be no acceleration in the expansion of the universe, i.e. if $V(t)$ is the volume of a small ball of test particles that start out at rest relative to each other, then $\lim_{V \to 0} V''(t) \leq 0$ [1].

But recent astronomical observations appear to show that the universe is not only expanding but also undergoing accelerated expansion [7,8]. So it is necessary to modify the timelike convergence condition $\mathrm{Ric}(v, v) \geq 0$. Moreover, we cannot assume that $\mathrm{Ric}(v, v)$ for any timelike unit vector v has a lower bound since M is not compact.

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We will obtain the volume expansion rate with the average of the negative part of the Ricci curvature. Let an observer move along a timelike geodesic Γ. Let

$$\rho(t) = \max\{0, -\mathrm{Ric}(\Gamma'(t), \Gamma'(t))\},$$

which is the negative part of the Ricci curvature along Γ. For a positive real number p, let

$$k_\Gamma(p, R) = \frac{1}{R}\int_0^R \rho(t)^p \mathrm{d}t.$$

Then k_Γ is an average of the negative part of the Ricci curvature along Γ. If $\mathrm{Ric}(v, v) \geq 0$ for any timelike vector v, then $k_\Gamma(p, R) = 0$ for any R and Γ.

Let $\Gamma(0) = x_0$ and H be the spacelike hyperspace H perpendicular to $\Gamma'(0)$ around $x_0 \in H$ at time t. Let $B(x_0, a)$ be the small ball in H around x_0 which is diffeomorphic to $\mathbb{R}^{n-1}$. Let particles at x in $B(x_0, a)$ start out at rest relative to each other and the timelike geodesic passing through x be Γ_x with $\Gamma_x(0) = x$. Then we consider the coordinate map $\phi : B(x_0, a) \times \mathbb{R} \to M$ such that $\phi(x, t) = \Gamma_x(t)$. Around $x_0 \in M$ and a spacelike hyperspace H which is perpendicular to $\Gamma'(0)$, we write the volume element as $\mathrm{dvol} = \omega \mathrm{d}t \wedge \mathrm{d}\theta_{n-1}$, where $\mathrm{d}\theta_{n-1}$ is the volume element on H. Then $\omega(x_0, t)\mathrm{d}\theta_{n-1}$ is the volume form of the spacelike hyperspace $\phi(B(x_0, a), t)$ perpendicular to $\Gamma'(t)$ at $\phi(x_0, t)$. Let V be the infinitesimal volume expansion rate at $t = 0$, i.e. $\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\, \omega = V\omega(0)$. Our main theorem is as follows.

Theorem 1. *Let M be an n-dimensional Lorentzian manifold. Then*

$$\frac{\omega(x_0, R)}{\omega(x_0, 0)} \leq \left(\frac{VR}{n-1} + 1\right)^{n-1} \exp(R\sqrt{n-1}\, k_\Gamma(p, R)^{\frac{1}{2p}}).$$

If $\lim_{R\to 0} k_\Gamma(p, R) = K$, *then*

$$\lim_{R\to\infty} \frac{\log \omega(x_0, R)}{R} \leq \sqrt{n-1}\, K^{\frac{1}{2p}}.$$

In particular, if $\lim_{R\to 0} k_\Gamma(p, R) = 0$, asymptotically the universe does not expand exponentially rapidly, i.e. for any $\epsilon > 0$,

$$\lim_{R\to\infty} \frac{\omega(x_0, R)}{\mathrm{e}^{\epsilon R}} = 0.$$

We do not need to assume $2p > n$, unlike [6].

In Riemannian geometry, for an n-dimensional compact Riemannian manifold M, we define the volume entropy $h(M)$ of M as follows:

$$h(M) = \lim_{R\to\infty} \frac{\log(\mathrm{vol}(B(x, R)))}{R},$$

where $B(x, R)$ is the R-ball centered at x in the universal covering space $\tilde{M}$ [5]. An upper bound of the volume entropy for negatively curved manifolds is obtained only with integrals of Ricci curvature over closed geodesics in M [5].

Note that in the Robertson–Walker model, $k_\Gamma(p, R)$ depends only on time, so we obtain an upper bound of the total volume expansion rate with $k_\Gamma(p, R)$ for only one timelike geodesic Γ from the above theorem.

2. Proof of main theorem

Let $c(s)$ be a curve from x_0 to $x \in B(x_0, a)$ in H. Let $J(t) = \frac{\partial}{\partial s}|_{s=0} \Gamma_{c(s)}(t)$; then $J(t)$ is a Jacobi field along the timelike geodesic $\Gamma(t)$. Let us write $J' = AJ$ for a linear transformation A depending on t. Then the Riccati equation is

$$A' + A^2 + R = 0$$

for the curvature operator R [2,3]. Hence we obtain that

$$\operatorname{tr} A' + \operatorname{tr} A^2 + \operatorname{Ric}(\Gamma', \Gamma') = 0. \tag{2.1}$$

We write

$$\omega' = \frac{\mathrm{d}}{\mathrm{d}t}\omega = h\omega. \tag{2.2}$$

Then $h = \operatorname{tr} A$, so h satisfies that

$$h' + \frac{h^2}{n-1} \leq -\operatorname{Ric}(\Gamma', \Gamma'), \tag{2.3}$$

from (2.1) [6].

If $\operatorname{Ric}(\Gamma', \Gamma') \geq 0$ for any $t \geq 0$, then

$$h' + \frac{h^2}{n-1} \leq 0, \tag{2.4}$$

which implies that

$$\left(\frac{1}{h}\right)' = -\frac{h'}{h^2} \geq \frac{1}{n-1}.$$

If $\omega'(0) = V\omega(0)$, then $h(0) = V$ and so

$$h(t) \leq \frac{n-1}{t + \frac{n-1}{V}}.$$

We denote $\frac{n-1}{t+\frac{n-1}{V}}$ by $h_0(t)$ and

$$h_0' + \frac{h_0^2}{n-1} = 0. \tag{2.5}$$

Let $\psi(t) = \max\{0, h(t) - h_0(t)\}$. Since $\frac{\omega'}{\omega} = h$, integrating this equation, we obtain that

$$\log\left(\frac{\omega(x_0, R)}{\omega(x_0, 0)}\right) = \int_0^R h\mathrm{d}t \leq \int_0^R h_0\mathrm{d}t + \int_0^R \psi\mathrm{d}t. \tag{2.6}$$

Then we have

$$\omega(x_0, R) \leq \mathrm{e}^{\int_0^R \psi\mathrm{d}t}\mathrm{e}^{\int_0^R h_0\mathrm{d}t}\omega(x_0, 0). \tag{2.7}$$

Integrating h_0, we obtain that

$$\int_0^R h_0\mathrm{d}t = (n-1)\log\left(\frac{VR}{n-1} + 1\right)$$

so

$$\mathrm{e}^{\int_0^R h_0\mathrm{d}t} \leq \left(\frac{VR}{n-1} + 1\right)^{n-1}. \tag{2.8}$$

Now it remains to calculate $\mathrm{e}^{\int_0^R \psi\mathrm{d}t}$.

We will prove the following lemma similarly to Lemma 2.2 in [6]:

Lemma 1. *For any $p > 0$, we have*

$$\int_0^r \psi^{2p}\mathrm{d}t \leq (n-1)^p \int_0^r \rho^p\mathrm{d}t.$$

Proof. From (2.3) and (2.5), we have

$$\psi' + \frac{\psi^2}{n-1} + \frac{2\psi h_0}{n-1} \le \rho \tag{2.9}$$

as we see in [6]. Multiply by ψ^{2p-2} and integrate to get

$$\int_0^r \psi'\psi^{2p-2}\mathrm{d}t + \frac{1}{n-1}\int_0^r \psi^{2p}\mathrm{d}t + \frac{2}{n-1}\int_0^r h_0\psi^{2p-1}\mathrm{d}t \le \int_0^r \rho\psi^{2p-2}\mathrm{d}t. \tag{2.10}$$

Since $(\psi^{2p-1})' = (2p-1)\psi'\psi^{2p-2}$,

$$\int_0^r \psi'\psi^{2p-2}\mathrm{d}t = \frac{1}{2p-1}\psi^{2p-1}\Big|_0^r \ge 0. \tag{2.11}$$

Inserting this in (2.11), we obtain that

$$\frac{1}{n-1}\int_0^r \psi^{2p}\mathrm{d}t + \frac{2}{n-1}\int_0^r h_0\psi^{2p-1}\mathrm{d}t \le \int_0^r \rho\psi^{2p-2}\mathrm{d}t, \tag{2.12}$$

which implies that

$$\begin{aligned} \frac{1}{n-1}\int_0^r \psi^{2p}\mathrm{d}t &\le \int_0^r \rho\psi^{2p-2}\mathrm{d}t \\ &\le \left(\int_0^r \rho^p\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^r \psi^{2p}\mathrm{d}t\right)^{1-\frac{1}{p}}. \end{aligned} \tag{2.13}$$

Dividing by $(\int_0^r \psi^{2p}\mathrm{d}t)^{1-\frac{1}{p}}$, then we obtain that

$$\frac{1}{n-1}\left(\int_0^r \psi^{2p}\mathrm{d}t\right)^{\frac{1}{p}} \le \left(\int_0^r \rho^p\mathrm{d}t\right)^{\frac{1}{p}}. \tag{2.14}$$

Consequently, we have

$$\int_0^r \psi^{2p}\mathrm{d}t \le (n-1)^p\int_0^r \rho^p\mathrm{d}t, \tag{2.15}$$

which completes the proof of Lemma 1. □

In the proof of Lemma 2.2 in [6], it is an essential condition that $p > n/2$, but our lemma holds for any positive p. By the Hölder inequality, we have

$$\begin{aligned} \frac{1}{R}\int_0^R \psi\mathrm{d}t &\le \frac{1}{R}\left(\int_0^R \psi^{2p}\mathrm{d}t\right)^{\frac{1}{2p}} R^{\frac{1}{q}} \\ &\le \frac{1}{R}\left((n-1)^p\int_0^R \rho^p\mathrm{d}t\right)^{\frac{1}{2p}} R^{\frac{1}{q}} \\ &\le \sqrt{n-1}\left(\frac{1}{R}\int_0^R \rho^p\mathrm{d}t\right)^{\frac{1}{2p}} \end{aligned} \tag{2.16}$$

for $1/2p + 1/q = 1$. We have

$$\begin{aligned} \frac{1}{R}\int_0^R \psi\mathrm{d}t &\le \sqrt{n-1}\left(\frac{1}{R}\int_0^R \rho^p\mathrm{d}t\right)^{\frac{1}{2p}} \\ &\le \sqrt{n-1}\,(k_\Gamma(p,R))^{\frac{1}{2p}}. \end{aligned} \tag{2.17}$$

Inserting (2.8) and (2.17) to (2.7), we obtain Theorem 1. □

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